Twisted K-theory and Poincaré duality

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Abstract

Using methods of KK-theory, we generalize Poincaré K-duality to the framework of twisted K-theory.

Key words: twisted K-theory, KK-theory, Poincaré duality, groupoid.

Introduction

In [4], Connes and Skandalis showed, using Kasparov's KK-theory, that given a compact manifold M, the K-theory of M is isomorphic to the K-homology of TM and vice-versa. It is well-known to experts that a similar result holds in twisted K-theory, although this is apparently written nowhere in the literature. In this paper, using Kasparov's more direct approach, we show that given any (graded) locally trivial bundle $\mathcal{A}$ of elementary C^*-algebras over M, the C^*-algebras of continuous sections $C(M, \mathcal{A})$ and $C(M, \mathcal{A}^{op} \hat{\otimes} \mathrm{Cliff}(TM \otimes \mathbb{C}))$ are K-dual to each other. When $\mathcal{A} = M$ is the trivial bundle, we recover Poincaré duality between $C(M)$ and $C_\tau(M) := C(M, \mathrm{Cliff}(TM \otimes \mathbb{C}))$ [8], which is equivalent to Poincaré duality between $C(M)$ and $C_0(TM)$ since $C_\tau(M)$ and $C_0(TM)$ are KK-equivalent to each other.

1 Preliminaries

In this paper, we will assume that the reader is familiar with the language of groupoids (although this is not crucial in the proof of the main theorem concerning Poincaré duality).

We just recall the definition of a generalized morphism (see e.g. [6]), since it is used at several places. Suppose that $G \rightrightarrows G^{(0)}$ and $\Gamma \rightrightarrows \Gamma^{(0)}$ are two Lie groupoids. Then a generalized morphism from G to Γ is given by a space P, two maps $G^{(0)} \overset{\tau}{\leftarrow} P \overset{\sigma}{\rightarrow} \Gamma^{(0)}$, a left action of G on P with respect to τ, a right action of Γ on P with respect to σ, such that the two actions commute, and $P \to G^{(0)}$ and is a right Γ-principal bundle. The set of isomorphism classes of generalized morphisms from G to Γ is denoted by $H^1(G, \Gamma)$. There is a category whose objects are Lie groupoids and arrows are isomorphism classes of generalized morphisms; isomorphisms in this category are called Morita equivalences.

If $f : G \to \Gamma$ is a map such that $f(gh) = f(g)f(h)$ whenever g and h are composable, then f is called a (strict) morphism. Then f determines a generalized morphism $P_f = G^{(0)} \times_{\Gamma^{(0)}} \Gamma$. Two strict morphisms f and f' determine the same element of $H^1(G, \Gamma)$ if there exists $\lambda : G^{(0)} \to \Gamma$ such that $f'(g) = \lambda(t(g)) f(g) \lambda(s(g))^{-1}$.

Finally, we recall that any element of $H^1(G, \Gamma)$ is given by the composition of a Morita equivalence with a strict morphism.

2 Graded twists and twisted K-theory

In this section, we review the basic theory of twisted K-theory in the graded setting, sometimes in more detail than some other references like [1, 5, 3]. This is a probably well-known and straightforward generalization of the ungraded case as developed e.g. in [1, 10], hence we will omit most proofs.

2.1 Graded Dixmier-Douady bundles

Let $\mathcal{M} \rightrightarrows \mathcal{M}^{(0)}$ be a Lie groupoid (more generally, most of the theory below is still valid for locally compact groupoids having a Haar system). The reader who is not interested in equivariant K-theory may assume that $\mathcal{M} = \mathcal{M}^{(0)} = M$ is just a compact manifold.

A *graded Dixmier-Douady bundle* of parity 0 (resp. of parity 1) $\mathcal{A}$ over $\mathcal{M} \rightrightarrows \mathcal{M}^{(0)}$ is a locally trivial bundle of $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebras over $\mathcal{M}^{(0)}$, endowed with a continuous action of $\mathcal{M}$, such that for all $x \in \mathcal{M}^{(0)}$, the fiber $\mathcal{A}_x$ is isomorphic to the $\mathbb{Z}/2\mathbb{Z}$-graded algebra $\mathcal{K}(\hat{H}_x)$ of compact operators over a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\hat{H}_x$ (resp. to the $\mathbb{Z}/2\mathbb{Z}$-graded algebra $\mathcal{K}(H_x) \oplus \mathcal{K}(H_x) \cong \mathcal{K}(H_x)\hat{\otimes}\mathbb{C}\ell_1$, where H_x is some Hilbert space and $\mathbb{C}\ell_1$ is the first complex Clifford algebra). Beware that H_x or $\hat{H}_x$ does not necessarily depend continuously on x. Of course, the action of $\mathcal{M}$ is required to preserve the degree. The usual theory of graded twists [1] corresponds to even graded D-D bundles (i.e. D-D bundles of parity 0), but our slightly more general definition allows to cover Clifford bundles as well: if $E \to M$ is a Euclidean vector bundle of dimension d, then $\mathrm{Cliff}(E\otimes\mathbb{C}) \to M$ is a graded D-D bundle of parity (d mod 2).

Denote by $\hat{H}$ the graded Hilbert space $H^0 \oplus H^1$, where $H^i = \ell^2(\mathbb{N})$, and $\hat{H}_{\mathcal{M}} = L^2(\mathcal{M}) \otimes \hat{H}$, where $L^2(\mathcal{M})$ is the $\mathcal{M}$-equivariant $\mathcal{M}^{(0)}$-Hilbert module obtained from $C_c(\mathcal{M})$ by completion with respect to the scalar product

$$\langle \xi, \eta\rangle(x) = \int\limits_{g\in\mathcal{M}^x} \overline{\xi(g)}\eta(g)\,\lambda^x(dg).$$

Two graded D-D bundles $\mathcal{A}$ and $\mathcal{A}'$ are said to be Morita equivalent if (they have the same parity and) $\mathcal{A}\hat{\otimes}\mathcal{K}(\hat{H}_{\mathcal{M}}) \cong \mathcal{A}'\hat{\otimes}\mathcal{K}(\hat{H}_{\mathcal{M}})$. The set of Morita equivalence classes of graded D-D bundles forms a group $\widehat{Br}_*(\mathcal{M}) = \widehat{Br}_0(\mathcal{M}) \oplus \widehat{Br}_1(\mathcal{M})$, the graded Brauer group of $\mathcal{M}$. The sum of $\mathcal{A}$ and $\mathcal{A}'$ is $\mathcal{A}\hat{\otimes}\mathcal{A}'$ (note that the parities do add up), and the opposite $\mathcal{A}^{op}$ of $\mathcal{A}$ is the bundle whose fibre at $x \in \mathcal{M}^{(0)}$ is the conjugate algebra of $\mathcal{A}_x$. In other words, $(\mathcal{A}^{op})_x = \mathcal{K}(\hat{H}_x^*)$ (resp. $(\mathcal{A}^{op})_x = \mathcal{K}(H_x^*) \oplus \mathcal{K}(H_x^*)$) in the even (resp. odd) case.

Moreover, $\sigma_i : \mathcal{A} \mapsto \mathcal{A}\hat{\otimes}\mathbb{C}\ell_1$ is an isomorphism from $\widehat{Br}_i(\mathcal{M})$ to $\widehat{Br}_{1-i}(\mathcal{M})$ such that $\sigma^2 = \mathrm{Id}$, hence $\widehat{Br}_*(\mathcal{M}) \cong \widehat{Br}_0(\mathcal{M}) \times \mathbb{Z}/2\mathbb{Z}$. Therefore, to study $\widehat{Br}_*(\mathcal{M})$ it suffices to study $\widehat{Br}_0(\mathcal{M})$.

Let us examine the relation between the graded Brauer group $\widehat{Br}_0(\mathcal{M})$ and bundles of projective unitary operators. Given any two graded Hilbert spaces $\hat{H}_1$ and $\hat{H}_2$, we denote by $\hat{U}(\hat{H}_1, \hat{H}_2)$ the set of unitary operators from $\hat{H}_1$ to $\hat{H}_2$ which are homogeneous of degree 0 or 1, and by $P\hat{U}(\hat{H}_1, \hat{H}_2)$ its quotient by S^1. When $\hat{H}_1 = \hat{H}_2$, these sets will be denoted by $\hat{U}(\hat{H}_1)$ and $P\hat{U}(\hat{H}_1)$.

The set $H^1(\mathcal{M}, P\hat{U}(\hat{H}))$ is actually an abelian monoid: given two generalized morphisms f_1 and f_2 from $\mathcal{M}$ to $P\hat{U}(\hat{H})$, the composition of $(f_1, f_2) : \mathcal{M} \to P\hat{U}(\hat{H}) \times P\hat{U}(\hat{H})$

with the morphism $(u,v)\mapsto u\hat{\otimes}v$ is a generalized morphism from $\mathcal{M}$ to $P\hat{U}(\hat{H}\hat{\otimes}\hat{H})\cong P\hat{U}(\hat{H})$.

Note that $H^1(\mathcal{M},\hat{U}(\hat{H}))$ is not a monoid, since given two morphisms $f_1,\, f_2:\Gamma\to\hat{U}(\hat{H})$ (with Γ Morita equivalent to $\mathcal{M}$), the map $f:g\mapsto f_1(g)\hat{\otimes}f_2(g)$ is not a morphism since

$$f(gh)=(-1)^{|f_2(g)|\,|f_1(h)|}f(g)f(h) \tag{1}$$

On the other hand, if we restrict to degree 0 operators, i.e. if we consider $\hat{U}(\hat{H})^0\cong U(H^0)\times U(H^1)$, then $H^1(\mathcal{M},\hat{U}(\hat{H})^0)$ is again a monoid.

The sequence

$$H^1(\mathcal{M},\hat{U}(\hat{H})^0)\to H^1(\mathcal{M},P\hat{U}(\hat{H}))\to\widehat{Br}_0(\mathcal{M})\to 0,$$

where the first map is the quotient map and the second is $P\mapsto P\times_{P\hat{U}(\hat{H})}\mathcal{K}(\hat{H})$, is canonically split-exact (the proof is analogue to [10]), and the splitting identifies $\widehat{Br}_0(\mathcal{M})$ with $H^1(\mathcal{M},P\hat{U}(\hat{H}))_{stable}=\{[P]|\;[P]=[P+P_0]\}$, where $P_0=P\hat{U}(L^2(\hat{H}_{\mathcal{M}}),\hat{H})$.

Let us recall the relation with the ordinary Brauer group $Br(\mathcal{M})$ of Morita equivalence classes of ungraded D-D bundles. Recall that $Br(\mathcal{M})\cong H^2(\mathcal{M}_\bullet,\underline{S^1})$, where $\underline{S^1}$ is the sheaf of smooth S^1-valued functions, and that $Br(\mathcal{M})\cong H^3(\mathcal{M}_\bullet,\mathbb{Z})$ when $\mathcal{M}\rightrightarrows\mathcal{M}^{(0)}$ is a proper groupoid, for instance, when $(\mathcal{M}\rightrightarrows\mathcal{M}^{(0)})=(M\times G\rightrightarrows M)$ is the crossed-product of a manifold by a proper action of a Lie group G.

There is a split exact sequence [1, 5]

$$0\to Br(\mathcal{M})\to\widehat{Br}_0(\mathcal{M})\to H^1(\mathcal{M},\mathbb{Z}/2\mathbb{Z})\to 0. \tag{2}$$

Indeed, from the exact sequence $1\to P\hat{U}(\hat{H})^0\to P\hat{U}(\hat{H})\to\mathbb{Z}/2\mathbb{Z}\to 0$, we get an exact sequence $0\to H^1(\mathcal{M},P\hat{U}(\hat{H})^0)_{stable}\to H^1(\mathcal{M},P\hat{U}(\hat{H}))_{stable}\to H^1(\mathcal{M},\mathbb{Z}/2\mathbb{Z})\to 0$. Moreover, there is an isomorphism $H^1(\mathcal{M},P\hat{U}(\hat{H})^0)_{stable}\cong H^2(\mathcal{M},\underline{S}^1)\cong Br(\mathcal{M})$ (this is analogue to the fact that $H^1(\mathcal{M},PU(H))\cong Br(\mathcal{M})$).

Furthermore, in the decomposition $\widehat{Br}_0(\mathcal{M})\cong H^1(\mathcal{M},\mathbb{Z}/2\mathbb{Z})\times Br(\mathcal{M})$, the sum becomes

$$(\delta_1,\mathcal{A}_1)+(\delta_2,\mathcal{A}_2)=(\delta_1+\delta_1,\mathcal{A}_1+\mathcal{A}_2+\delta_1\cdot\delta_2),$$

where $\delta_1\cdot\delta_2$ is the element of $H^2(\mathcal{M},\underline{S}^1)$ corresponding to the cocycle $(\delta_1\cdot\delta_2)(g,h)=(-1)^{\delta_2(g)\delta_1(h)}$. This can be seen by direct checking using (1) (see [1, Proposition 2.3] for a different explanation).

2.2 Graded S^1-central extensions

Definition 2.1 A graded S^1-central extension of a groupoid $\Gamma\rightrightarrows\Gamma^{(0)}$ is a central extension $S^1\to\widetilde{\Gamma}\xrightarrow{\pi}\Gamma$, together with a groupoid morphism $\delta:\Gamma\to\mathbb{Z}/2\mathbb{Z}$.

One defines the sum of two graded central extensions $(\widetilde{\Gamma}_1,\delta_1)$ and $(\widetilde{\Gamma_2},\delta_2)$ as $(\widetilde{\Gamma},\delta)$, where $\delta(g)=\delta_1(g)+\delta_2(g)$ and $\widetilde{\Gamma}=(\widetilde{\Gamma}_1\times_\Gamma\widetilde{\Gamma}_2)/S^1=\{(g_1,g_2)\in\widetilde{\Gamma}_1\times\widetilde{\Gamma}_2|\;\pi_1(g_1)=\pi_2(g_2)\}/\sim$, where $\sim$ is the equivalence relation $(g_1,g_2)\sim(g_1',g_2')\iff\exists\lambda\in S^1,\;(g_1',g_2')=(\lambda g_1,\lambda^{-1}g_2)$.

The multiplication for the groupoid $\widetilde{\Gamma}$ is $(\widetilde{g}_1,\widetilde{g}_2)(\widetilde{h}_1,\widetilde{h}_2)=(-1)^{\delta_2(g)\delta_1(h)}(\widetilde{g}_1\widetilde{h}_1,\widetilde{g}_2\widetilde{h}_2)$, where $g=\pi_i(\widetilde{g}_i)$, $h=\pi_i(\widetilde{h}_i)$.

Note that the set of isomorphism classes of graded S^1-central extensions of Γ forms an abelian group. To see that the product is commutative, if $\widetilde{\Gamma}' = (\widetilde{\Gamma}_2 \times_\Gamma \widetilde{\Gamma}_1)/S^1$ is endowed with the product $(\widetilde{g}_2, \widetilde{g}_1)(\widetilde{h}_2, \widetilde{h}_1) = (-1)^{\delta_1(g)\delta_2(h)}(\widetilde{g}_2\widetilde{h}_2, \widetilde{g}_1\widetilde{h}_1)$, then

$$\begin{array}{ccc} \widetilde{\Gamma} & \to & \widetilde{\Gamma}' \\ (\widetilde{g}_1, \widetilde{g}_2) & \mapsto & (-1)^{\delta_1(g)\delta_2(g)}(\widetilde{g}_2, \widetilde{g}_1) \end{array}$$

is a S^1-equivariant isomorphism.

To see that $(\widetilde{\Gamma}, \delta)$ has an inverse, let $\widetilde{\Gamma}^{op}$ be equal to $\widetilde{\Gamma}$ as a set, but the S^1-principal bundle structure is replaced by the conjugate one, and the product $*_{op}$ in $\widetilde{\Gamma}^{op}$ is

$$\widetilde{g} *_{op} \widetilde{h} = (-1)^{\delta(g)\delta(h)}\widetilde{g}\widetilde{h}.$$

Then

$$\begin{array}{ccc} \Gamma \times S^1 & \to & (\widetilde{\Gamma} \times_\Gamma \widetilde{\Gamma}^{op})/S^1 \\ (g, \lambda) & \mapsto & [\lambda\widetilde{g}, \widetilde{g}] \end{array}$$

is an isomorphism ($\widetilde{g} \in \widetilde{\Gamma}$ is any lift of $g \in \Gamma$).

Let us define the group $\widehat{\mathrm{Ext}}(\mathcal{M}, S^1)$. Consider the collection of triples $(S^1 \to \widetilde{\Gamma} \to \Gamma, \delta, P)$ where $(S^1 \to \widetilde{\Gamma} \to \Gamma, \delta)$ is a graded central extension and P is a Morita equivalence from $\Gamma \to \mathcal{M}$. Two such triples $(S^1 \to \widetilde{\Gamma}_1 \to \Gamma_1, \delta_1, P_1)$ and $(S^1 \to \widetilde{\Gamma}_2 \to \Gamma_2, \delta_2, P_2)$ are said to be Morita equivalent if there exists a Morita equivalence $\widetilde{Q} : \widetilde{\Gamma}_1 \to \widetilde{\Gamma}_2$ which is S^1-equivariant, such that the diagrams of isomorphism classes of generalized morphisms

$$\begin{tikzcd} \Gamma_1 \arrow[r, "{[Q]}"] \arrow[dr, "{[P_1]}"'] & \Gamma_2 \arrow[d, "{[P_2]}"] \\ & \mathcal{M} \end{tikzcd}$$

and

$$\begin{tikzcd} \Gamma_1 \arrow[r, "{[Q]}"] \arrow[dr, "{[\delta_1]}"'] & \Gamma_2 \arrow[d, "{[\delta_2]}"] \\ & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

commute, where $Q : \Gamma_1 \to \Gamma_2$ is the Morita equivalence induced by $\widetilde{Q}$. Then the group $\widehat{\mathrm{Ext}}(\mathcal{M}, S^1)$ is the quotient of the collection of triples by Morita equivalence.

Then $\widehat{Br}_0(\mathcal{M}) \cong \widehat{\mathrm{Ext}}(\mathcal{M}, S^1)$. Let us explain the map $\widehat{Br}_0(\mathcal{M}) \to \widehat{\mathrm{Ext}}(\mathcal{M}, S^1)$.

Any element of $\widehat{Br}_0(\mathcal{M})$ is given by a bundle $\mathcal{A}$ with fiber $\mathcal{K}(\hat{H})$, hence by a $P\hat{U}(\hat{H})$-principal bundle P over $\mathcal{M} \rightrightarrows \mathcal{M}^{(0)}$, i.e. by a generalized morphism $f : \mathcal{M} \to P\hat{U}(\hat{H})$. Let $\mathcal{E}$ be the graded central extension $(S^1 \to \hat{U}(\hat{H}) \to P\hat{U}(\hat{H}), \delta)$, where $\delta : P\hat{U}(\hat{H}) \to \mathbb{Z}/2\mathbb{Z}$ is the degree map. Then $f^*\mathcal{E}$ is an element of $\widehat{\mathrm{Ext}}(\mathcal{M}, S^1)$.

Remark 2.2 Let $f_1^*\mathcal{E}$, $f_2^*\mathcal{E} \in \widehat{\mathrm{Ext}}(\mathcal{M}, S^1)$. Then the sum of $f_1^*\mathcal{E}$ and $f_2^*\mathcal{E}$ (where $f_i : \Gamma \to P\hat{U}(\hat{H})$ is a strict morphism and $\Gamma \rightrightarrows \Gamma^{(0)}$ is a groupoid which is Morita-equivalent to $\mathcal{M}$) is given by $f^*\mathcal{E}$, where

$$\begin{array}{ccc} f : \Gamma & \to & P\hat{U}(\hat{H}\hat{\otimes}\hat{H}) \cong P\hat{U}(\hat{H}) \\ g & \mapsto & f_1(g)\hat{\otimes} f_2(g). \end{array}$$

(Note that f is indeed a homomorphism, since $f(gh) = (-1)^{\delta(g)\delta(h)} f(g)f(h)$ agrees with $f(g)f(h)$ up to a scalar in S^1.)

2.3 Twisted K-theory

Let $\mathcal{A} \to \mathcal{M}^{(0)}$ be a graded D-D bundle over $\mathcal{M} \rightrightarrows \mathcal{M}^{(0)}$. We define $K^*_{\mathcal{A}}(\mathcal{M})$ as $K_*(\mathcal{A} \rtimes_r \mathcal{M})$, the K-theory of the reduced crossed-product of the graded C^*-algebra $\mathcal{A}$ by the action of $\mathcal{M}$. If $\mathcal{A}$ corresponds to the graded central extension $(S^1 \to \widetilde{\Gamma} \to \Gamma, \delta)$, then $K^*_{\mathcal{A}}(\mathcal{M})$ is isomorphic to $K_*(C^*_r(\widetilde{\Gamma})^{S^1})$, where the C^*-algebra $C^*_r(\widetilde{\Gamma})^{S^1}$ is subalgebra of $C^*_r(\widetilde{\Gamma})$ which is the closure of

$$\{f \in C_c(\widetilde{\Gamma}) |\; f(\lambda g) = \lambda^{-1} f(g) \quad \forall g \in \widetilde{\Gamma}\; \forall \lambda \in S^1\}.$$

The C^*-algebra $C^*_r(\widetilde{\Gamma})^{S^1}$ is considered as a $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebra, using the grading automorphism

$$f \in C_c(\widetilde{\Gamma}) \mapsto (\gamma \mapsto f(\gamma)\delta(\gamma)) \in C_c(\widetilde{\Gamma}).$$

Note that it suffices to study $K^0_{\mathcal{A}}(\mathcal{M})$, since $K^1_{\mathcal{A}}(\mathcal{M}) = K^0_{\mathcal{A}\hat{\otimes}\mathbb{C}\ell_1}(\mathcal{M})$.

2.4 Example of manifolds

Let M be a manifold. Elements of $\widehat{\operatorname{Ext}}(M, S^1)$ are given by an open cover $(U_i)_{i\in I}$, smooth maps $c_{ijk} : U_{ijk} = U_i \cap U_j \cap U_k \to S^1$ and $\delta_{ij} : U_{ij} \to \mathbb{Z}/2\mathbb{Z}$ such that $\delta_{ij} + \delta_{jk} = \delta_{ik}$ and $c_{jkl}c^{-1}_{ikl}c_{ijl}c^{-1}_{ijk} = 1$.

Let $\Gamma = \amalg_{i,j} U_{ij}$ and $\widetilde{\Gamma} = \Gamma \times S^1$. Define a product on $\widetilde{\Gamma}$ by $(x_{ij}, \lambda)(x_{jk}, \mu) = (x_{ik}, \lambda\mu c_{ijk})$. Then $\widetilde{\Gamma}$ is a groupoid, and there is a central extension $S^1 \to \widetilde{\Gamma} \to \Gamma$.

The sum of (c^1, δ^1) and (c^2, δ^2) is (c, δ) where $\delta_{ij} = \delta^1_{ij} + \delta^2_{ij}$ and $c_{ijk} = c^1_{ijk} c^2_{ijk} (-1)^{\delta^2_{ij}\delta^1_{jk}}$.

Let us consider the particular case when $c = 1$ is the trivial cocycle. In that case, $C^*_r(\widetilde{\Gamma})^{S^1} \cong C^*_r(\Gamma)$. Let us compare this $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebra to the $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebra $C_0(\widetilde{M})$, where $\widetilde{M} \to M$ is the double cover determined by the cocycle δ. Let $P = (\amalg U_i) \times_M \widetilde{M}$. Then P is a $\mathbb{Z}/2\mathbb{Z}$-equivariant Morita equivalence from Γ to $\widetilde{M} \rtimes \mathbb{Z}/2\mathbb{Z}$, hence $K^*_{\mathcal{A}}(M) = K_*(C^*_r(\widetilde{\Gamma})^{S^1}) = K_*(C^*_r(\Gamma)) = K_*(C_0(\widetilde{M}) \rtimes \mathbb{Z}/2\mathbb{Z}) = K_*(C_0(\widetilde{M})\hat{\otimes}\mathbb{C}\ell_1) = K_{*+1}(C_0(\widetilde{M}))$ as in [5, Remark A.13].

2.5 Twistings by Euclidean vector bundles

Suppose that E is a Euclidean vector bundle over $\mathcal{M} \rightrightarrows \mathcal{M}^{(0)}$. Then E is given by an $O(n)$-principal bundle over $\mathcal{M} \rightrightarrows \mathcal{M}^{(0)}$, hence by a morphism $f : \Gamma \to O(n)$ together with a Morita equivalence from Γ to $\mathcal{M}$.

Let $\mathcal{E}$ be the graded S^1-central extension

$$S^1 \to Pin^c(n) \to O(n),$$

where $Pin^c(n) = Pin(n) \times_{\{\pm 1\}} S^1$, and $\delta : O(n) \to \mathbb{Z}/2\mathbb{Z}$ is the map such that $\det A = (-1)^{\delta(A)}$. Then $f^*\mathcal{E}$ is a graded central extension of Γ, hence determines an even graded D-D bundle $\mathcal{A}_E$.

On the other hand, $\mathcal{A}'_E = \operatorname{Cliff}(E \otimes_{\mathbb{R}} \mathbb{C}) \to \mathcal{M}^{(0)}$ is another graded D-D bundle over $\mathcal{M} \rightrightarrows \mathcal{M}^{(0)}$ which has the same parity as $\dim E$. We want to compare it to $\mathcal{A}_E$.

We first need two lemmas.

Lemma 2.3 *Let G be a compact Lie group. Denote by G_0 its identity component. Assume that G_0 is simply connected and that $Br(G/G_0) = \{0\}$. Then every central extension of G by S^1 is split.*

Proof. Since G is compact, every S^1-central extension is of finite order. Let us recall the argument: given a central extension $\mathcal{E} = (S^1 \to \widetilde{G} \to G)$, let V be a finite dimensional representation of $\widetilde{G}$ which is a sub-representation of $\{f \in L^2(\widetilde{G}) |\ f(\lambda g) = \lambda^{-1} f(g)\ \forall \lambda \in S^1,\ \forall g \in \widetilde{G}\}$. Let $d = \dim V$, $n = d!$ and $W = \Lambda^d V \cong \mathbb{C}$. Then the representation of $\widetilde{G}$ in W is a map $\widetilde{G} \to U(W) \cong S^1$ which is a splitting of $n\,\mathcal{E}$, hence $\mathcal{E}$ is of order at most n.

Therefore, the extension $\mathcal{E}$ comes from a central extension $0 \to \mathbb{Z}/n\mathbb{Z} \to \widetilde{G} \to G \to 1$. Since G is simply connected, the central extension must be trivial as $\mathbb{Z}/n\mathbb{Z}$-principal bundle, i.e. $\widetilde{G} = G \times \mathbb{Z}/n\mathbb{Z}$, and the product on $\widetilde{G}$ is given by $(g, \lambda)(h, \mu) = (gh, \lambda + \mu + c(g, h))$ where $c : G \times G \to \mathbb{Z}/n\mathbb{Z}$ is a 2-cocycle. Using connectedness of G_0, c must factor through $G/G_0 \times G/G_0$, i.e. the central extension is pulled back from a central extension of G/G_0, which must be trivial by assumption. □

Lemma 2.4 *Let G be a Lie group and G_0 a normal subgroup containing the identity component of G such that G_0 has no nontrivial character and that $Br(G_0) = \{0\}$.*

If $\mathcal{E}$ and $\mathcal{E}'$ are S^1-central extensions whose restriction to G_0 are isomorphic, then $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic.

Proof. After taking the difference of $\mathcal{E}$ and $\mathcal{E}'$, we may assume that $\mathcal{E}'$ is the trivial extension. Denote by $S^1 \to \widetilde{G} \to G$ the extension $\mathcal{E}$. Let $g \mapsto \widetilde{g}$ be a splitting $G_0 \to \widetilde{G}$. Choose a family (s_i) such that $G = \amalg_i s_i G_0$, and for each i, choose a lift $\widetilde{s}_i$ of s_i. Define then $\widetilde{s_i g}$ by $\widetilde{s}_i \widetilde{g}$. By construction, $\widetilde{\gamma h} = \widetilde{\gamma}\widetilde{h}$ for all $(\gamma, h) \in G \times G_0$.

Next, define the 2-cocycle $c : G \times G \to S^1$ by $\widetilde{g}\widetilde{h} = c(g, h)\widetilde{gh}$. Let $c_{ij} = c(s_i, s_j)$.

For all j, let $\varphi_j : G_0 \to S^1$ such that $\widetilde{s}_j^{-1}\widetilde{g}\widetilde{s}_j = \varphi_j(g)\widetilde{s_j^{-1} g s_j}$. It is immediate to check that φ_j is a group morphism, hence φ_j is trivial by assumption, i.e. $\widetilde{s}_j^{-1}\widetilde{g}\widetilde{s}_j = \widetilde{s_j^{-1} g s_j}$. Multiplying on the right by $\widetilde{h}$ and on the left by $\widetilde{s}_i\widetilde{s}_j$, we get $\widetilde{s}_i\widetilde{g}\widetilde{s}_j\widetilde{h} = \widetilde{s}_i\widetilde{s}_j\widetilde{s_j^{-1} g s_j h} = c_{ij}\widetilde{s_i s_j s_j^{-1} g s_j h} = c_{ij}\widetilde{s_i g s_j h}$, hence $\widetilde{s_i g}\widetilde{s_j h} = c_{ij}\widetilde{s_i g s_j h}$. It follows that $c(s_i g, s_j h) = c_{ij}$, i.e. that c factors through $G/G_0 \times G/G_0$. Since $Br(G/G_0)$ is trivial by assumption, c must be a coboundary. We conclude that $\mathcal{E}$ is a split extension. □

Proposition 2.5 $K_{*+\dim E, \mathcal{A}_E}(\mathcal{M}) = K_{*, \mathcal{A}'_E}(\mathcal{M}) = K_*(C_0(\mathcal{M}^{(0)}, \mathrm{Cliff}(E \otimes_{\mathbb{R}} \mathbb{C})) \rtimes_r \mathcal{M}) = K_*(C_0(E) \rtimes_r \mathcal{M})$.

Proof. The last equality follows from the fact that $C_0(E)$ and $C_0(\mathcal{M}^{(0)}, \mathrm{Cliff}(E \otimes_{\mathbb{R}} \mathbb{C}))$ are $\mathcal{M}$-equivariantly KK-equivalent [7]. The second equality is just the definition.

To prove the first equality, let us suppose for instance that $n = \dim E$ is even, the proof for n odd being analogous. We have to compare the graded D-D bundle $\mathcal{A}'_E$ with the graded central extension $S^1 \to \widetilde{\Gamma} \to \Gamma$ which is pulled back from $S^1 \to Pin^c(n) \to O(n)$. By naturality, we can just assume that $\Gamma = \mathcal{M} = O(n)$, and that $E = \mathbb{R}^n$ is endowed with the canonical action of $O(n)$.

Then, $\mathrm{Cliff}(E \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{C}\ell_n = \mathcal{L}(\hat{\mathcal{H}})$, where $\hat{\mathcal{H}}$ is the graded Hilbert space $\mathbb{C}^{2^{n/2-1}} \oplus \mathbb{C}^{2^{n/2-1}}$.

Denote by $\alpha : O(n) \to P\hat{U}(\hat{\mathcal{H}})$ the canonical action of $O(n)$ on $\mathbb{C}\ell_n = \mathrm{Cliff}(E_{\mathbb{C}})$. To show that the central extension associated to the graded D-D bundle $\mathrm{Cliff}(E_{\mathbb{C}}) \to \cdot$ is $(S^1 \to Pin^c(n) \to O(n))$, it suffices to prove that there exists a lifting $\widetilde{\alpha}$:

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & S^1 \arrow[d] \\
Pin^c(n) \arrow[r, "\widetilde{\alpha}"] \arrow[d] & \hat{U}(\hat{\mathcal{H}}) \arrow[d] \\
O(n) \arrow[r, "\alpha"] & P\hat{U}(\hat{\mathcal{H}}).
\end{tikzcd}$$

For $n = 2$ it is elementary to check that both vertical lines are split extensions. For $n \geq 3$, since $Spin(n)$ is simply connected and compact, it has no nontrivial S^1-central extension (see Lemma 2.3), hence the pull-back of $\mathcal{E} = (S^1 \to \hat{U}(\hat{\mathcal{H}}) \to P\hat{U}(\hat{H}))$ by the map $Spin(n) \to O(n)$ has a lift $\beta : Spin(n) \to \hat{U}(\hat{\mathcal{H}})$.

If $\beta(-1) = Id$ then β induces a map $\bar{\beta} : SO(n) \to \hat{U}(\hat{\mathcal{H}})$, which means that the extension $S^1 \to Spin^c(n) \to SO(n)$ is split. It follows that $S^1 \to Spin^c(3) \to SO(3)$ is split, i.e. that there exists a morphism $\varphi : Spin^c(3) = SU(2) \times_{\{\pm 1\}} S^1 \to S^1$ such that $\varphi(\lambda g) = \lambda \varphi(g)$ for all $\lambda \in S^1$ and all $g \in SU(2) \times_{\{\pm 1\}} S^1$. Putting $\chi(g) = \varphi(g, 1)$, the morphism $\chi : SU(2) \to S^1$ satisfies $\chi(-1) = -1$. Using simplicity of $SU(2)/\{\pm 1\} = SO(3)$, it follows that χ is injective, which is impossible.

It follows that $\beta(-1) = -Id$, hence β induces a lift $\beta : Spin^c(n) \to \hat{U}(\hat{\mathcal{H}})$ which is S^1-equivariant. This means that the restriction of $\mathcal{E}$ to $SO(n)$ is isomorphic to $S^1 \to Spin^c(n) \to SO(n)$. To conclude that the restriction of $\mathcal{E}$ to $O(n)$ is isomorphic to $S^1 \to Pin^c(n) \to O(n)$, we apply Lemma 2.4 to $G = O(n)$ and $G_0 = SO(n)$. $\Box$

3 Poincaré duality

3.1 Kasparov's constructions

Let M be a compact manifold (actually, Poincaré duality can be generalized to arbitrary manifolds [8], but in this paper we confine ourselves to compact ones for simplicity). We suppose that M is endowed with a Riemannian metric which is invariant by the action of a locally compact group G. Given any vector bundle $\mathcal{A}$ over any manifold M, we denote by $C_{\mathcal{A}}(M)$ the space of continuous sections vanishing at infinity. We will also write $C_{\mathcal{A}}$ whenever there is no ambiguity. We denote by τ the complexified cotangent bundle of M.

In [8], Kasparov constructed two elements

$$\theta \in KK_{M \rtimes G}(C(M), C(M) \otimes C_\tau(M)) = RKK_G(M; \mathbb{C}, C_\tau(M))$$

and $D \in KK_G(C_\tau(M), \mathbb{C})$ (in this paper, we will use Le Gall's [9] notation $KK_{M \rtimes G}(\cdot, \cdot)$ for equivariant KK-theory with respect to the groupoid $M \rtimes G$, rather than Kasparov's $RKK_G(M; \cdot, \cdot)$, but of course both are equivalent).

Let us recall the construction of θ and D.

Let $H = L^2(\Lambda^* M)$, and

$$\begin{array}{rcl} \varphi : C_\tau(M) & \to & \mathcal{L}(H) \\ \omega & \mapsto & e(\omega) + e(\omega)^*, \end{array}$$

where $e(\omega)$ is the exterior multiplication, and let $F = \mathcal{D}(1+\mathcal{D})^{-1/2}$ where $\mathcal{D} = d + d^*$. Then $D = [(H, \varphi, F)]$.

Let us explain the construction of θ. Denoting by ρ the distance function on M, let $r > 0$ be so small that for all (x, y) in $U = \{(x, y) \in M \times M | \ \rho(x, y) < r\}$, there exists a unique geodesic from x to y.

For every $C(M \times M)$-algebra A, we denote by A_U the C^*-algebra $C_0(U)A$. Then the element θ is defined as $[(C_M \otimes C_\tau(M))_U, \Theta]$ where $\Theta = (\Theta_x)_{x \in M}$, $\Theta_x(y) = \frac{\rho(x,y)}{r}(d_y\rho)(x, y) \in T^*_y M \subset \mathrm{Cliff}_{\mathbb{C}}(T^*_y M)$.

3.2 Constructions in twisted K-theory

In this subsection, we construct an element $\theta^{\mathcal{A}} \in KK_{M \rtimes G}(C(M), C_{\mathcal{A}} \hat{\otimes} C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}})$ for any graded D-D bundle $\mathcal{A}$ over $(M \times G \rightrightarrows M)$, i.e. for any G-equivariant graded D-D bundle over M. We may assume that $\mathcal{A}$ is stabilized, i.e. that $\mathcal{A} \cong \mathcal{A} \hat{\otimes} \mathcal{K}(\hat{H} \otimes L^2(G))$. First, let us denote by $p_t(x, y)$ the geodesic segment joining x to y at constant speed $(0 \leq t \leq 1)$.

Using p_t, we see that $p_t : U \to M$ is a G-equivariant homotopy equivalence. Unfortunately, this does not imply that $Br(U \rtimes G)$ and $Br(M \rtimes G)$ are isomorphic for arbitrary G, hence we will make the following

Assumption. In the sequel of this paper, and unless stated otherwise, G will be a *compact* Lie group acting smoothly on a compact manifold M.

In that case, $H^2(U \rtimes G, \underline{S}^1) \cong H^3(U \rtimes G, \mathbb{Z}) = H^3(\frac{U \times EG}{G}, \mathbb{Z}) \cong H^3(\frac{M \times EG}{G}, \mathbb{Z}) \cong H^2(M \rtimes G, \underline{S}^1)$. As a consequence, there is a continuous, G-equivariant family of isomorphisms

$$u_{t,x,y} : \mathcal{A}_x \xrightarrow{\sim} \mathcal{A}_{p_t(x,y)}.$$

Of course, the u_t's are not unique, but this will not be important as far as K-theory is concerned as we will see.

Consider the canonical Morita-equivalence $\mathcal{H}_x$ between $\mathbb{C}$ and $\mathcal{A}_x \hat{\otimes} \mathcal{A}_x^{op}$. Let $\mathcal{H} = (\mathcal{H}_x)_{x \in M}$ be the corresponding Morita equivalence between $C(M)$ and $C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}}(M)$. Then $[((C(M) \otimes C_\tau(M))_U \hat{\otimes}_{C(M)} \mathcal{H}, \Theta \hat{\otimes} 1)]$ is an element of $KK_{M \rtimes G}(C(M), C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}}(M) \hat{\otimes} C_\tau(M))$.

Now, using the map $u_{1,x,y} : \mathcal{A}_x \xrightarrow{\sim} \mathcal{A}_y$, we get a Morita equivalence $\mathcal{E}_{x,y}$ from $\mathcal{A}_x^{op}$ to $\mathcal{A}_y^{op}$, thus a Morita equivalence $\mathcal{E}$ from $(C_{\mathcal{A}^{op}}(M) \hat{\otimes} C(M))_U$ to $(C(M) \hat{\otimes} C_{\mathcal{A}^{op}}(M))_U$. Tensoring over $C(M \times M)$ with $C_{\mathcal{A}}(M) \hat{\otimes} C_\tau(M)$, we get a Morita equivalence $\mathcal{E}' = (\mathcal{E}'_{x,y})_{(x,y) \in U}$ from $(C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}}(M) \hat{\otimes} C_\tau(M))_U$ to $(C_{\mathcal{A}}(M) \hat{\otimes} C_{\tau \hat{\otimes} \mathcal{A}^{op}}(M))_U$.

We then define $\theta^{\mathcal{A}}$ as

$$\begin{aligned} \theta^{\mathcal{A}} &= [((C(M) \otimes C_\tau(M))_U \hat{\otimes}_{C(M)} \mathcal{H} \hat{\otimes}_{C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_\tau} \mathcal{E}', \Theta \hat{\otimes} 1)] \\ &\in KK_{M \rtimes G}(C(M), C_{\mathcal{A}}(M) \hat{\otimes} C_{\tau \hat{\otimes} \mathcal{A}^{op}}(M)). \end{aligned}$$

3.3 Twisted K-homology

Given a C^*-algebra A endowed with an action of a locally compact group G, the G-equivariant K-homology of A, $K^*_G(A)$, is defined by $KK^*_G(A, \mathbb{C})$. If $\mathcal{A}$ is a G-equivariant graded D-D bundle over M, we define $K^{G,\mathcal{A}}_*(M)$ by $K^*_G(C_{\mathcal{A}}(M))$.

3.4 Maps between twisted K-theory and twisted K-homology

Let A and B be two separable graded G-C^*-algebras (recall that G is assumed to be a compact Lie group). We define two maps

$$\begin{aligned} \mu : KK_G(A, C_{\mathcal{A}}(M)\hat{\otimes}B) &\to KK_G(C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}A, B) \\ \nu : KK_G(C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}A, B) &\to KK_G(A, C_{\mathcal{A}}(M)\hat{\otimes}B). \end{aligned}$$

First, let us introduce some notations. Suppose that M is a locally compact space endowed with an action of a locally compact group. If A, B and D are G-equivariant graded $C(M)$-algebras, and $\mathcal{E}$ is a A-B-C^*-bimodule, then Kasparov defined a $A\hat{\otimes}_{C_0(M)}D$-$B\hat{\otimes}_{C_0(M)}D$-C^*-bimodule $\sigma_{M,D}(\mathcal{E})$, and thus a "suspension" map $\sigma_{M,D} : KK_{M\rtimes G}(A, B) \to KK_{M\rtimes G}(A\hat{\otimes}_{C_0(M)}D, B\hat{\otimes}_{C_0(M)}D)$. There is also a suspension map $\sigma_D : KK_{M\rtimes G}(A, B) \to KK_{M\rtimes G}(A\hat{\otimes}D, B\hat{\otimes}D)$ defined in a similar way.

Given a $(A_1, B_1\hat{\otimes}D)$- C^*-bimodule $\mathcal{E}_1$ and a $(D\hat{\otimes}A_2, B_2)$-C^*-bimodule $\mathcal{E}_2$, the $(A_1\hat{\otimes}A_2, B_1\hat{\otimes}B_2)$-$C^*$-bimodule $\mathcal{E}_1\hat{\otimes}_D\mathcal{E}_2$ is defined by $\sigma_{A_2}(\mathcal{E}_1)\hat{\otimes}_{A_2\hat{\otimes}B_1}\sigma_{B_1}(\mathcal{E}_2)$.

We introduce a similar notation when all tensor products are replaced by tensor products over a space M: given a $(A_1, B_1\hat{\otimes}_{C(M)}D)$- C^*-bimodule $\mathcal{E}_1$ and every $(D\hat{\otimes}_{C(M)}A_2, B_2)$-$C^*$-bimodule $\mathcal{E}_2$, $\mathcal{E}_1\hat{\otimes}_D\mathcal{E}_2$ is a $(A_1\hat{\otimes}_{C(M)}A_2, B_1\hat{\otimes}_{C(M)}B_2)$-$C^*$-bimodule.

Let us now define μ and ν. First, let us note that $KK_G(A, C_{\mathcal{A}}(M)\hat{\otimes}B)$ is isomorphic to $KK_{M\rtimes G}(C(M)\otimes A, C_{\mathcal{A}}(M)\hat{\otimes}B)$.

The map μ is defined as the composition

$$\begin{aligned} KK_{M\rtimes G}(C(M)\otimes A, C_{\mathcal{A}}(M)\hat{\otimes}B) &\overset{\sigma_{M,C_{\tau\hat{\otimes}\mathcal{A}^{op}}}}{\to} KK_{M\rtimes G}(C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}\mathcal{A}, C_{\tau\hat{\otimes}\mathcal{A}^{op}\hat{\otimes}\mathcal{A}}\hat{\otimes}B) \\ &\overset{\cdot\hat{\otimes}\mathcal{H}^{op}}{\to} KK_{M\rtimes G}(C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}A, C_\tau\hat{\otimes}B) \\ &\overset{\otimes D}{\to} KK_G(C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}A, B). \end{aligned}$$

The map ν is just $\theta^{\mathcal{A}}\otimes\cdot : KK_G(C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}A, B) \to KK_{M\rtimes G}(C(M)\otimes A, C_{\mathcal{A}}\hat{\otimes}B)$.

3.5 The main theorem

Theorem 3.1 *Let G be a compact Lie group acting on a compact manifold M, and let A and B be two graded separable G-C^*-algebras. Let $\mathcal{A}$ be a G-equivariant graded D-D bundle over M.*

Then the maps μ and ν defined above are inverse to each other:

$$KK_G^*(A, C_{\mathcal{A}}(M)\hat{\otimes}B) \cong KK_G^*(C_{\tau\hat{\otimes}\mathcal{A}^{op}}(M)\hat{\otimes}A, B).$$

Replacing $\mathcal{A}$ by $\tau\hat{\otimes}\mathcal{A}^{op}$, we get:

$$KK_G^*(C_{\mathcal{A}}(M)\hat{\otimes}A, B) \cong KK_G^*(A, C_{\tau\hat{\otimes}\mathcal{A}^{op}}(M)\hat{\otimes}B).$$

In particular, for $A = B = \mathbb{C}$ we get

$$\begin{aligned} K_{G,\mathcal{A}}^*(M) &\cong K_*^{G,\tau\hat{\otimes}\mathcal{A}^{op}}(M) \\ K_*^{G,\mathcal{A}}(M) &\cong K_{G,\tau\hat{\otimes}\mathcal{A}^{op}}^*(M). \end{aligned}$$

Remark 3.2 This result (in the case when G is the trivial group) is observed in [2, Section 7].

Remark 3.3 The map μ does not depend of the choice of the isomorphisms $u_{t,x,y}$, hence ν doesn't either.

The rest of the paper is devoted to the proof of Theorem 3.1.

3.6 Proof of $\mu \circ \nu = \mathbf{Id}$

For all $\alpha \in KK_G(C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A, B)$, we have

$$\mu \circ \nu(\alpha) = \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^{\mathcal{A}}) \hat{\otimes}_{C_{\mathcal{A}^{op} \hat{\otimes} \mathcal{A}}} \mathcal{H}^{op} \otimes_{C_\tau} D \otimes_{C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A} \alpha.$$

Thus, we need to prove that

$$\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^{\mathcal{A}}) \hat{\otimes}_{C_{\mathcal{A}^{op} \hat{\otimes} \mathcal{A}}} \mathcal{H}^{op} \otimes_{C_\tau} D = 1 \in KK_G(C_{\tau \hat{\otimes} \mathcal{A}^{op}}, C_{\tau \hat{\otimes} \mathcal{A}^{op}}).$$

Consider the element $\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta) \in KK_{M \rtimes G}(C_{\tau \hat{\otimes} \mathcal{A}^{op}}, C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_\tau)$. Denote by

$$\begin{array}{rcl} s : C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_\tau & \overset{\cong}{\to} & C_\tau \hat{\otimes} C_{\tau \hat{\otimes} \mathcal{A}^{op}} \\ x \otimes y & \mapsto & (-1)^{\deg x . \deg y} y \otimes x \end{array}$$

the flip. Suppose proven that

$$\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta) \otimes [s] = \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^{\mathcal{A}}) \hat{\otimes}_{C_{\mathcal{A}^{op} \hat{\otimes} \mathcal{A}}} \mathcal{H}^{op}. \tag{3}$$

Then

$$\begin{array}{rcl} \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^{\mathcal{A}}) \otimes_{C_{\mathcal{A}^{op} \hat{\otimes} \mathcal{A}}} \mathcal{H}^{op} \otimes_{C_\tau} D & = & \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta) \otimes_{C_\tau} D \\ & = & \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta \otimes_{C_\tau} D) = \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(1) = 1 \end{array}$$

Since $\theta \otimes_{C_\tau} D = 1$ (from [8, Theorem 4.8]).

We postpone the proof of (3) until subsection 3.8.

3.7 Proof of $\nu \circ \mu = \mathbf{Id}$

For all $\alpha \in KK_{M \rtimes G}(C(M) \otimes A, C_{\mathcal{A}}(M) \hat{\otimes} B)$, we have

$$\nu \circ \mu(\alpha) = \theta^{\mathcal{A}} \otimes_{C_{\tau \hat{\otimes} \mathcal{A}^{op}}} (\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\alpha) \otimes_{C_{\mathcal{A}^{op} \otimes \mathcal{A}}} \mathcal{H}^{op} \otimes_{C_\tau} D).$$

Suppose shown that

$$\begin{array}{rcl} \theta^{\mathcal{A}} \otimes_{C_{\tau \hat{\otimes} \mathcal{A}^{op}}} \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\alpha) \otimes_{C_{\mathcal{A}^{op} \hat{\otimes} \mathcal{A}}} \mathcal{H}^{op} & = & \alpha \otimes_{C_{\mathcal{A}}} \sigma_{M, C_{\mathcal{A}}}(\theta) \\ & & \in KK_{M \rtimes G}(C(M) \otimes A, C_{\mathcal{A}} \hat{\otimes} C_\tau \hat{\otimes} B). \end{array} \tag{4}$$

Then $\nu \circ \mu(\alpha) = \alpha \otimes_{C_{\mathcal{A}}} (\sigma_{M, C_{\mathcal{A}}}(\theta) \otimes_{C_\tau} D) = \alpha \otimes_{C_{\mathcal{A}}} \sigma_{M, C_{\mathcal{A}}}(\theta \otimes_{C_\tau} D) = \alpha \otimes_{C_{\mathcal{A}}} 1 = \alpha$.

We postpone the proof of (b) until subsection 3.9

3.8 Proof of (3)

Recall the proof when $\mathcal{A}$ is the trivial bundle [8, Lemma 4.6]. We want to show that $\sigma_{M,C_\tau}(\theta)$ is flip-invariant. Denote by $p_t^* : T^*_{p_t(x,y)}M \hookrightarrow T^*_{(x,y)}U$ the pull-back map induced by p_t, and let q_t^* be the isometry $q_t^* = p_t^*(p_t p_t^*)^{-1/2}$. We denote again by $q_t^* : \Omega^1(M) \hookrightarrow \Omega^1(U)$ the corresponding map. Then q_t^* induces a map $\varphi_t : C_\tau(M) \to \mathcal{L}(C_\tau(U))$.

Let $\beta(s) = (p_{t(1-s)}(x,y), p_{t(1-s)+s}(x,y)) \in U$, and

$$\Theta_t(x,y) = \frac{\rho(x,y)}{r} \left\| \frac{d\beta}{ds}|_{s=1} \right\|^{-1} \frac{d\beta}{ds}|_{s=1} \in T_{x,y}U.$$

Then $(C_\tau(U), \varphi_t, \Theta_t)_{0\leq t\leq 1}$ is a homotopy between $\sigma_{M,C_\tau}(\theta)$ and $\sigma_{M,C_\tau}(\theta)\otimes[s]$.

Now, consider the general case. $\sigma_{M,C_{\tau\hat{\otimes}\mathcal{A}^{op}}}(\theta)$ is the Kasparov triple

$$((C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}C_\tau)_U, \varphi, \Theta)$$

where $\varphi : C_{\tau\hat{\otimes}\mathcal{A}^{op}} \to \mathcal{L}((C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}C_\tau)_U)$ is the obvious map. Thus,

$$\sigma_{M,C_{\tau\hat{\otimes}\mathcal{A}^{op}}}(\theta)\otimes[s] = [((C_\tau\hat{\otimes}C_{\tau\hat{\otimes}\mathcal{A}^{op}})_U, \varphi, \Theta_1)]$$

with $\theta_1 = \frac{\rho(x,y)}{r}(d_x\rho)(x,y) \in T_x^*M \subset \mathrm{Cliff}_{\mathbb{C}}(T_x^*M)$, while

$$\sigma_{M,C_{\tau\hat{\otimes}\mathcal{A}^{op}}}(\theta^{\mathcal{A}}) \otimes_{C_{\mathcal{A}^{op}\hat{\otimes}\mathcal{A}}} \mathcal{H}^{op} = [((C_{\tau\hat{\otimes}\mathcal{A}^{op}}(M)\hat{\otimes}C_\tau(M))_U\hat{\otimes}_{C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}C_\tau}\mathcal{E}'', \varphi', \Theta\otimes 1)]$$

where $\mathcal{E}''$ is the Morita equivalence between $(C_{\tau\hat{\otimes}\mathcal{A}^{op}}\hat{\otimes}C_\tau)_U$ and $(C_\tau\hat{\otimes}C_{\tau\hat{\otimes}\mathcal{A}^{op}})_U$ obtained from the Morita equivalence $\mathcal{E}$ between $p_0^*C_{\mathcal{A}^{op}} = (C_{\mathcal{A}^{op}}\otimes C(M))_U$ and $p_1^*C_{\mathcal{A}^{op}} = (C(M)\otimes C_{\mathcal{A}^{op}})_U$.

Let $\mathcal{E}_t = (\mathcal{E}_{x,y,t})_{(x,y)\in U}$ be the Morita equivalence between $p_t^*C_{\mathcal{A}^{op}}$ and $p_1^*C_{\mathcal{A}^{op}}$ constructed in the same way as $\mathcal{E} = \mathcal{E}_0$.

Then

$$\begin{aligned}
\sigma_{M,C_{\tau\hat{\otimes}\mathcal{A}^{op}}}(\theta)\otimes[s] &= (C_\tau(U)\hat{\otimes}_{C_0(U)}\mathcal{E}_1, \varphi, \Theta_1)\\
\sigma_{M,C_{\tau\hat{\otimes}\mathcal{A}^{op}}}(\theta^{\mathcal{A}})\hat{\otimes}_{C_{\mathcal{A}^{op}\hat{\otimes}\mathcal{A}}}\mathcal{H}^{op} &= (C_\tau(U)\hat{\otimes}_{C_0(U)}\mathcal{E}_0, \varphi', \Theta).
\end{aligned}$$

Let Θ_t as above. We produce a homotopy

$$(C_\tau(U)\hat{\otimes}_{C_0(U)}\mathcal{E}_t, \psi_t, \Theta_t)$$

between those two elements. Only $\psi_t : C_{\tau\hat{\otimes}\mathcal{A}^{op}} \to \mathcal{L}(C_\tau(U)\hat{\otimes}_{C_0(U)}\mathcal{E}_t)$ remains to be defined. We need two compatible maps

$$\begin{aligned}
\psi_t' : C_\tau &\to \mathcal{L}(C_\tau(U)\hat{\otimes}_{C_0(U)}\mathcal{E}_t)\\
\text{and } \psi_t'' : C_{\mathcal{A}^{op}} &\to \mathcal{L}(C_\tau(U)\hat{\otimes}_{C_0(U)}\mathcal{E}_t).
\end{aligned}$$

The map ψ_t' is just $\varphi_t\otimes 1$. The map ψ_t'' is given by the composition

$$C_{\mathcal{A}^{op}} \xrightarrow{p_t^*} C_b(U, p_t^*\mathcal{A}^{op}) \to \mathcal{L}(\mathcal{E}_t).$$

3.9 Proof of (4)

Let us first recall the proof when $\mathcal{A}$ is trivial [8, Lemma 4.5]. We want to show that for all $\alpha \in KK_{M\rtimes G}(C(M)\otimes A, C(M)\otimes B)$ we have

$$\alpha \otimes_{C(M)} \theta = \theta \otimes_{C_\tau(M)} \sigma_{M,C_\tau(M)}(\alpha) \in KK_{M\rtimes G}(C(M)\otimes A, C(M)\otimes C_\tau(M)\hat{\otimes} B).$$

Write $\alpha = [(E,T)]$ where $\overline{C(M,A)E} = E$ and T is G-continuous. Then both products can be written as

$$[(E\hat{\otimes}_{C(M)}(C(M)\otimes C_\tau(M))_U, \varphi_i, F_i)]$$

where F_i is of the form $M_2^{1/2}(T\hat{\otimes}1) + M_1^{1/2}(1\hat{\otimes}\Theta)$ $(i=0,1)$, and where the map $C(M)\to \mathcal{L}((C(M)\otimes C_\tau(M))_U)$ used to define φ_i is p_i^*.

Since p_0 and p_1 are homotopic, φ_0 and φ_1 are homotopic. One then constructs a homotopy between F_0 and F_1 using Kasparov's technical theorem as in [8, Lemma 4.5].

Let us now consider a general G-equivariant graded D-D bundle $\mathcal{A}$ over M. Let $\alpha = [(E,T)] \in KK_{M\rtimes G}(C(M)\otimes A, C_{\mathcal{A}}\hat{\otimes} B)$ where $\overline{C(M,A)\,E} = E$ and T G-continuous.

We want to show that $\alpha \otimes_{C_{\mathcal{A}}} \sigma_{M,C_{\mathcal{A}}}(\theta) = \theta^{\mathcal{A}}\hat{\otimes}_{C_\tau\hat{\otimes}\mathcal{A}^{op}}(\alpha)\hat{\otimes}_{C_{\mathcal{A}^{op}\hat{\otimes}\mathcal{A}}}\mathcal{H}^{op}$. Let us just explain the homotopy between the two modules, the homotopy between the F_i's being obtained using Kasparov's technical theorem in the same way as in [8, Lemma 4.5].

The left-hand side is

$$E\hat{\otimes}_{C_{\mathcal{A}}}(C_{\mathcal{A}}\hat{\otimes} C_\tau)_U, \tag{5}$$

and the right-hand side is

$$E\hat{\otimes}_{C_{\mathcal{A}}}\sigma_{M,C_{\mathcal{A}}}(\mathcal{F}_1)\hat{\otimes}_{C_0(U)}\mathcal{H}'_1\hat{\otimes}_{C_{p_1^*(\mathcal{A}\hat{\otimes}\mathcal{A}^{op})}(U)}p_1^*\mathcal{H}^{op}, \tag{6}$$

where we recall that $p_1 : U \to M$ is the second projection $(x,y)\mapsto y$. The $C(M)$-$(C(M)\otimes C_\tau)_U$-bimodule $\mathcal{F}_1$ is $(C(M)\otimes C_\tau)_U$, with the left action of $C(M)$ on $\mathcal{F}_1$ obtained via $C(M) \xrightarrow{p_1^*} C_b(U) \to \mathcal{L}((C(M)\otimes C_\tau)_U)$.

$\mathcal{H}'_1$ is the Morita equivalence between $C_0(U)$ and $p_0^* C_{\mathcal{A}}\hat{\otimes}_{C_0(U)} p_1^* C_{\mathcal{A}^{op}}$ obtained by composing the Morita equivalence $p_0^*\mathcal{H}$ between $C_0(U)$ and $p_0^*(C_{\mathcal{A}\hat{\otimes}\mathcal{A}^{op}})$ with the isomorphism $p_0^*\mathcal{A}^{op} \cong p_1^*\mathcal{A}^{op}$.

Using the map $p_t : U \to M$ instead of p_1, consider (with obvious notations) the homotopy $E\hat{\otimes}_{C_{\mathcal{A}}}\sigma_{M,C_{\mathcal{A}}}(\mathcal{F}_t)\hat{\otimes}_{C_0(U)}\mathcal{H}'_t\hat{\otimes}_{C_{p_t^*(\mathcal{A}\hat{\otimes}\mathcal{A}^{op})}(U)}p_t^*\mathcal{H}^{op}$.

For $t=1$, we get (6).

For $t=0$, we get $E\hat{\otimes}_{C_{\mathcal{A}}}(C_{\mathcal{A}}\hat{\otimes} C_\tau)_U\hat{\otimes}_{C_0(U)}p_0^*\mathcal{H}\hat{\otimes}_{C_{p_0^*(\mathcal{A}\hat{\otimes}\mathcal{A}^{op})}(U)}p_0^*\mathcal{H}^{op}$ where the right $C_{p_0^*(\mathcal{A}\hat{\otimes}\mathcal{A}^{op})}(U)$-structure on $E\hat{\otimes}_{C_{\mathcal{A}}}(C_{\mathcal{A}}\hat{\otimes} C_\tau)_U\hat{\otimes}_{C_0(U)}p_0^*\mathcal{H}$ is defined as follows: $C_{p_0^*\mathcal{A}}$ acts on $(C_{\mathcal{A}}\hat{\otimes} C_\tau)_U$ by the obvious action, and $C_{p_0^*\mathcal{A}^{op}}$ acts on $p_0^*\mathcal{H}$. In other words, it is the tensor product of (5) with $\beta_{\mathcal{A}}$ over $C_{\mathcal{A}}$, where $\beta_{\mathcal{A}}$ is the $C_{\mathcal{A}}$-$C_{\mathcal{A}}$-bimodule

$$\beta_{\mathcal{A}} = (C_{\mathcal{A}}\hat{\otimes}_{C(M)}\mathcal{H})\hat{\otimes}_{C_{\mathcal{A}\hat{\otimes}\mathcal{A}^{op}}}\mathcal{H}^{op}.$$

In the expression above, the right $C_{\mathcal{A}\hat{\otimes}\mathcal{A}^{op}}$-module structure on $C_{\mathcal{A}}\hat{\otimes}_{C(M)}\mathcal{H}$ is defined as follows: $\forall a\in C_{\mathcal{A}}$, $\forall b \in C_{\mathcal{A}^{op}}$ $\forall \xi\otimes\eta \in C_{\mathcal{A}}\hat{\otimes}_{C(M)}\mathcal{H}$,

$$(\xi\otimes\eta)\cdot(a\otimes b) = (-1)^{|\eta|\,|a|}\xi a\otimes \eta b.$$

To finish the proof, it remains to show that $\beta_{\mathcal{A}} \cong C_{\mathcal{A}}$. Suppose for instance that $\mathcal{A}$ is an even graded D-D bundle. Let $x\in M$. Denoting by $\hat{H}_x$ a Hilbert space such that

$\mathcal{A}_x \cong \mathcal{K}(\hat{H}_x)$, we have $(\beta_{\mathcal{A}})_x = [\mathcal{K}(\hat{H}_x)\hat{\otimes}(\hat{H}_x^*\hat{\otimes}\hat{H}_x)]\hat{\otimes}_{\mathcal{A}_x\hat{\otimes}\mathcal{A}_x^{op}}(\hat{H}_x\hat{\otimes}\hat{H}_x^*)$, where in $\hat{H}_x\hat{\otimes}\hat{H}_x^*$, $\hat{H}_x$ (resp. $\hat{H}_x^*$) is considered as a $\mathcal{A}_x$-$\mathbb{C}$ (resp. a $\mathcal{A}_x^{op}$-$\mathbb{C}$-bimodule, and in $\hat{H}_x^*\hat{\otimes}\hat{H}_x$, $\hat{H}_x$ (resp. $\hat{H}_x^*$) is considered as a $\mathbb{C}$-$\mathcal{A}_x^{op}$ (resp. a $\mathbb{C}$-$\mathcal{A}_x$-bimodule, and the right $\mathcal{A}_x\hat{\otimes}\mathcal{A}_x^{op}$-module structure on $\mathcal{K}(\hat{H}_x)\hat{\otimes}(\hat{H}_x^*\hat{\otimes}\hat{H}_x)$ is $(\xi\otimes(\eta\otimes\zeta))\cdot(a\otimes b) = (-1)^{|a|\,(|\xi|+|\eta|)}\xi a\otimes\eta\otimes\zeta b$. It follows that $(\beta_{\mathcal{A}})_x \cong \hat{H}_x\hat{\otimes}\hat{H}_x^*$ is the natural $\mathcal{K}(\hat{H}_x)$-bimodule $\mathcal{K}(\hat{H}_x)$.

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